

n -CORRELATION WITH RESTRICTED SUPPORT

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ABSTRACT. We give a new proof that Rudnick and Sarnak's calculation of the n -correlation of zeros of L -functions matches random matrix theory. We do this using the derivation of n -correlation for eigenvalues of random unitary matrices from our previous paper. There we detailed a method using ratios of characteristic polynomials that is far less elegant than standard random matrix techniques, but which has the advantage of translating easily into the number theory case by applying the Ratios Conjectures for ratios of L -functions. The formulae in our previous paper are unwieldy in comparison with the standard determinantal random matrix version, but we show here that their form allows immediate identification of which terms remain when restrictions are placed on the support of the test function. After this it is straightforward to show that the surviving terms match Rudnick and Sarnak's result.

1. INTRODUCTION

Let $U(N)$ denote the group of $N \times N$ unitary matrices and let dU signify the Haar measure for this group. For a unitary matrix U we let $\theta_1, \dots, \theta_N$ denote the N eigenangles, i.e. $e^{i\theta_j}$ are the eigenvalues. A beautiful theorem (see [Meh]) in random matrix theory gives the n -correlation of these eigenvalues:

Theorem 1. *Let $F : [0, 2\pi]^n \rightarrow \mathbb{C}$ be an integrable function of n -variables. Then*

$$\int_{U(N)} \sum_{(j_1, \dots, j_n)}^* F(\theta_{j_1}, \dots, \theta_{j_n}) dU = \frac{1}{(2\pi)^n} \int_{[0, 2\pi]^n} F(\alpha_1, \dots, \alpha_n) \det_{n \times n} S_N(\alpha_k - \alpha_j) d\alpha_1 \dots d\alpha_n,$$

where \sum^* indicates that the sum is over n -tuples of distinct indices (j_1, \dots, j_n) with $1 \leq j_i \leq n$, i.e. $j_i \neq j_k$ if $i \neq k$, and where

$$(1) \quad S_N(\alpha) = \frac{\sin \frac{N\alpha}{2}}{\sin \frac{\alpha}{2}}.$$

In our previous article [CoSn] we give a different derivation for the n -correlation for the eigenvalues of matrices in $U(N)$ that is far less elegant, but which can be copied step by step to obtain the n -correlation of zeros of L -functions subject to the Ratios Conjecture.

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It is of interest to ask about

$$(2) \quad \int_{U(N)_{(j_1, \dots, j_n)}} \sum^* f\left(\frac{N\theta_{j_1}}{2\pi}, \dots, \frac{N\theta_{j_n}}{2\pi}\right) dU$$

and the scaled limit of this quantity as $N \rightarrow \infty$ in the situation where f is a translation invariant function that has a Fourier transform with limited support. Such a quantity arises in number theory, for example in the calculation by Rudnick and Sarnak [RuSa] of the n -correlation of the zeros of the Riemann zeta-function. The inherent difficulty of dealing with off-diagonal terms in the mean-value theorems which arise necessitate the limited support of \hat{f} . These off-diagonal terms lead to the consideration of the distribution of prime pairs in short intervals. See the work of Bogomolny and Keating [BoKe] for a beautiful demonstration that the combinatorics of prime number sums arising from the Hardy-Littlewood prime pair conjectures lead to the determinantal formula of random matrix theory.

In [RuSa], after deriving an expression for the n -correlation in the situation that their test function has a Fourier transform with support limited to a proper subset of $[-1, 1]$, the authors are faced with the non-trivial task of verifying that their answer agrees with theorems from random matrix theory. A similar problem occurs in two works on the n -level density of zeros of quadratic L-functions, first considered by Mike Rubinstein [Rub], and subsequently by Peng Gao in his thesis. In the former case Rubinstein finds, as do Rudnick and Sarnak, an adhoc method to verify the consistency of the number theory and the random matrix theory calculations. Peng Gao is unable to make this verification in his situation. A subsequent paper [ERR] rectifies this situation and completes the verification in Gao's case by a very clever appeal to zeta-functions over function fields.

In this paper, we show that the formula for n -correlation derived in [CoSn] allows for immediate identification of the terms which survive a restriction on the support of the test function, and so makes the verification of Rudnick and Sarnak's formula in the unitary case straightforward, eg. for the n -correlation of the zeros of the Riemann zeta-function. (In [RuSa] more general L-functions are considered; here we focus on the case of the Riemann zeta-function, but higher degree L-functions don't present any extra difficulty.) Our proof is natural in that it explains the situation for a test function whose Fourier transform has any range of support. The extension of our method to orthogonal and symplectic ensembles will be part of Amy Mason's thesis.

2. STATEMENT OF RESULTS

For simplicity we state the results of Rudnick and Sarnak for the n -correlation of the zeros $1/2 + i\gamma_j$ of the Riemann zeta-function, though there is no difficulty working in their full generality. Before doing so, we introduce their vector notation as a convenient way to express the combinatorial sum that arises in their work. Let

$$(3) \quad \begin{cases} \mathbf{e}_{i,j} &= \mathbf{e}_i - \mathbf{e}_j \\ \mathbf{e}_i &= (0, \dots, 1, \dots, 0) \text{ the } i\text{th standard basis vector} \end{cases}$$

Theorem 2. [RuSa] Theorem 3.1. Let h_j , for $1 \leq j \leq n$, be rapidly decaying functions with

$$(4) \quad h_j(x) = \int_{\mathbf{R}} g_j(t) e^{ixt} dt$$

where g_j is smooth and compactly supported. Suppose that

$$(5) \quad f(x_1, \dots, x_n) = \int_{\mathbf{R}^n} \Phi(\xi_1, \dots, \xi_n) \delta(\xi_1 + \dots + \xi_n) e(-x_1 \xi_1 - \dots - x_n \xi_n) d\xi_1 \dots d\xi_n$$

where Φ is smooth, even, and compactly supported in such a way that

$$(6) \quad \Phi(\xi_1, \dots, \xi_n) = 0$$

whenever

$$(7) \quad |\xi_1| + \dots + |\xi_n| > 2 - \epsilon$$

for some fixed $\epsilon > 0$. Let $\mathcal{L} = \log T$. Then

$$\begin{aligned} & \sum_{\gamma_1, \dots, \gamma_n} h_1\left(\frac{\gamma_1}{T}\right) \dots h_n\left(\frac{\gamma_n}{T}\right) f\left(\frac{\mathcal{L}\gamma_1}{2\pi}, \dots, \frac{\mathcal{L}\gamma_n}{2\pi}\right) \\ &= \kappa(\mathbf{h}) \frac{T\mathcal{L}}{2\pi} \left(\Phi(0) + \sum_{r=1}^{[n/2]} \sum_{\substack{i(t) < j(t) \\ t \leq r}} \int |v_1| \dots |v_r| \Phi(v_1 \mathbf{e}_{i(1), j(1)} + \dots + v_r \mathbf{e}_{i(r), j(r)}) dv \right) + O(T) \end{aligned}$$

where the sum is over all disjoint pairs of indices $i(t) < j(t)$ in $\{1, 2, \dots, n\}$ when $t \leq r$ and where

$$(8) \quad \kappa(\mathbf{h}) = \int_{\mathbf{R}} h_1(u) \dots h_n(u) du.$$

The main result of this paper is the random matrix analogue of this theorem. For the purposes of making a close connection with number theory it is convenient to let the eigenangles “wrap around.” Thus, for an $N \times N$ unitary matrix X we will let its eigenangles be

$$(9) \quad \dots \leq \theta_{-R} \leq \theta_{-R+1} \leq \dots \leq \theta_0 \leq \theta_1 \leq \dots \leq \theta_R \leq \dots$$

where

$$(10) \quad \theta_{r+kN} = \theta_r + 2\pi k.$$

Now we have infinitely many eigenangles just as for the Riemann zeta-function we have infinitely many zeros. We are interested in the sum

$$(11) \quad \sum_{j_1, \dots, j_n} h_1\left(\frac{\theta_{j_1}}{T}\right) \dots h_n\left(\frac{\theta_{j_n}}{T}\right) f\left(\frac{N\theta_{j_1}}{2\pi}, \dots, \frac{N\theta_{j_n}}{2\pi}\right)$$

where each j_k now runs over all integers. This sum exactly parallels the above sum over zeta-zeros.

Theorem 3. *With the same conditions on h and f , we have*

$$(12) \quad \int_{U(N)} \sum_{j_1, \dots, j_n} h_1 \left(\frac{\theta_{j_1}}{T} \right) \dots h_n \left(\frac{\theta_{j_n}}{T} \right) f \left(\frac{N\theta_{j_1}}{2\pi}, \dots, \frac{N\theta_{j_n}}{2\pi} \right) dU \\ = \kappa(\mathbf{h}) \frac{NT}{2\pi} \sum_{\substack{K+L+M=\{1, \dots, n\} \\ |K|=|L|}} \sum_{\sigma \in S_K} \int_{\xi_{k_j} > 0} \xi_{k_1} \dots \xi_{k_K} \Phi \left(\sum_{j=1}^K \xi_{k_j} \mathbf{e}_{k_j, \ell_{\sigma(j)}} \right) d\xi + O(N).$$

This theorem is slightly more general than the theorem above of Rudnick and Sarnak and has a smaller error term; but the main terms exactly match when Φ is assumed to be even as in [RuSa] and N takes the place of $\log T$. As a consequence, we have proven that the result of Rudnick and Sarnak agrees with random matrix theory without going through the complex combinatorial considerations that they undergo in the second half of their paper.

In general, one could ask to compare number theory and random matrix theory to see what happens if one works with a set of test functions \mathcal{T}_q which are defined as above except that the support of Φ is restricted so that

$$(13) \quad \Phi(\xi_1, \dots, \xi_n) = 0$$

whenever

$$(14) \quad |\xi_1| + \dots + |\xi_n| > 2q - \epsilon.$$

We have developed an approach in [CoSn] that treats number theory and random matrix theory in parallel. From this work it is a simple matter to read off the results when dealing with test functions from \mathcal{T}_q . This result is described below.

3. EIGENVALUE CORRELATIONS

Before stating the theorem of [CoSn] we describe some notation. We let

$$(15) \quad z(x) = \frac{1}{1 - e^{-x}}.$$

In our formulas for averages of characteristic polynomials the function $z(x)$ plays the role for random matrix theory that $\zeta(1+x)$ plays in the theory of moments of the Riemann zeta-function.

Given finite sets A and B we will have a sum over subsets $S \subset A$ and $T \subset B$ with $|S| = |T|$. We let $\overline{S} = A - S$ and $\overline{T} = B - T$. We will let $\hat{\alpha}$ denote a generic member of S and $\hat{\beta}$ denote a generic member of T ; we will use α and β for generic members of A and B or of \overline{S} and \overline{T} , according to the context. Also $S^- = \{-\hat{\alpha} : \hat{\alpha} \in S\}$, and similarly for T^- . We let

$$(16) \quad Z(A, B) := \prod_{\substack{\alpha \in A \\ \beta \in B}} z(\alpha + \beta).$$

A simple modification of Theorem 4 of [CoSn] yields

Theorem 4. *Let $\delta > 0$ and let $\int_{(c)}$ denote an integration along the vertical path from $c - i\infty$ to $c + i\infty$. Suppose that $F(x_1, \dots, x_n)$ is a holomorphic function which decays rapidly in each variable in vertical strips. Then, for any $\delta > 0$,*

$$(17) \quad \begin{aligned} & \int_{U(N)} \sum_{j_1, \dots, j_n} F(\theta_{j_1}, \dots, \theta_{j_n}) dX \\ &= \frac{1}{(2\pi i)^n} \sum_{K+L+M=\{1, \dots, n\}} (-1)^{|L|+|M|} N^{|M|} \\ & \quad \times \int_{(\delta)^{|K|}} \int_{(-\delta)^{|L|}} \int_{(0)^{|M|}} J^*(z_K; -z_L) F(iz_1, \dots, iz_n) dz_1 \dots dz_n \end{aligned}$$

where $z_K = \{z_k : k \in K\}$, $-z_L = \{-z_\ell : \ell \in L\}$ and $\int_{(\delta)^{|K|}} \int_{(-\delta)^{|L|}} \int_{(0)^{|M|}}$ means that we are integrating all the variables in z_K along the (δ) path, all of the variables in z_L along the $(-\delta)$ path and all of the variables in z_M along the (0) path; and

$$(18) \quad J^*(A, B) := \sum_{\substack{S \subset A, T \subset B \\ |S|=|T|}} e^{-N(\sum_{\hat{\alpha} \in S} \hat{\alpha} + \sum_{\hat{\beta} \in T} \hat{\beta})} \frac{Z(S, T) Z(S^-, T^-)}{Z^\dagger(S, S^-) Z^\dagger(T, T^-)} \sum_{\substack{(A-S)+(B-T) \\ = U_1 + \dots + U_Q \\ |U_q| \leq 2}} \prod_{q=1}^Q H_{S, T}(U_q),$$

where

$$(19) \quad H_{S, T}(W) = \begin{cases} \sum_{\hat{\alpha} \in S} \frac{z'}{z}(\alpha - \hat{\alpha}) - \sum_{\hat{\beta} \in T} \frac{z'}{z}(\alpha + \hat{\beta}) & \text{if } W = \{\alpha\} \subset A - S \\ \sum_{\hat{\beta} \in T} \frac{z'}{z}(\beta - \hat{\beta}) - \sum_{\hat{\alpha} \in S} \frac{z'}{z}(\beta + \hat{\alpha}) & \text{if } W = \{\beta\} \subset B - T \\ \left(\frac{z'}{z}\right)'(\alpha + \beta) & \text{if } W = \{\alpha, \beta\} \text{ with } \begin{smallmatrix} \alpha \in A-S, \\ \beta \in B-T \end{smallmatrix} \\ 0 & \text{otherwise.} \end{cases}$$

Also, $Z(A, B) = \prod_{\substack{\alpha \in A \\ \beta \in B}} z(\alpha + \beta)$, with the dagger imposing the additional restriction that a factor $z(x)$ is omitted if its argument is zero.

We remark that, assuming the ratios conjecture, see [CoSn], there is a structurally identical formula giving the n -correlation of the zeros of the Riemann zeta-function.

We want to apply the above but with

$$(20) \quad F(x_1, \dots, x_n) = f\left(\frac{Nx_1}{2\pi}, \dots, \frac{Nx_n}{2\pi}\right) h_1\left(\frac{x_1}{T}\right) \dots h_n\left(\frac{x_n}{T}\right)$$

with $f \in \mathcal{T}_q$.

In the right-hand side of the formula we replace f by its Fourier transform so that we can better see what the implications of limited support are. Thus, we write

$$f\left(\frac{iNz_1}{2\pi}, \dots, \frac{iNz_n}{2\pi}\right) = \int_{\mathbf{R}^n} \Phi(\xi_1, \dots, \xi_n) \delta(\xi_1 + \dots + \xi_n) e\left(-\frac{iNz_1\xi_1}{2\pi} - \dots - \frac{iNz_n\xi_n}{2\pi}\right) d\xi_1 \dots d\xi_n.$$

Observe that

$$(21) \quad e \left(-\frac{iNz_1\xi_1}{2\pi} - \dots - \frac{iNz_n\xi_n}{2\pi} \right) = e^{Nz_1\xi_1 + \dots + Nz_n\xi_n}.$$

For each i , $|\operatorname{Re} z_i| \leq \delta$. Suppose that $\Phi(\xi_1, \dots, \xi_n) = 0$ if $|\xi_1| + \dots + |\xi_n| > 2q - \epsilon$ for some $\epsilon > 0$. Then $|\Phi| \neq 0$ implies that

$$(22) \quad |e^{Nz_1\xi_1 + \dots + Nz_n\xi_n}| \leq e^{N\delta(2q-\epsilon)}.$$

We compare this exponential with the exponentials which appears in the factor $J^*(z_K; -z_L)$:

$$(23) \quad \sum_{\substack{S \subset z_K, T \subset -z_L \\ |S|=|T|}} e^{-N(\sum_{\hat{\alpha} \in S} \hat{\alpha} + \sum_{\hat{\beta} \in T} \hat{\beta})}.$$

Notice that the real parts of all of the $\alpha \in z_K$ and all of the $\beta \in -z_L$ are equal to δ . If $|S| = |T| \geq q$, then

$$(24) \quad \left| e^{-N(\sum_{\hat{\alpha} \in S} \hat{\alpha} + \sum_{\hat{\beta} \in T} \hat{\beta})} \right| \leq e^{-2N\delta q}.$$

Thus, the product of these two factors is $\leq e^{-N\delta\epsilon}$. We can move the paths of integration $(-\delta)$ and (δ) away from the imaginary axis. The integrand tends to zero uniformly on the vertical paths as $\delta \rightarrow \infty$. Thus, we see that these terms are 0.

We let $J_q^*(A; B)$ be defined as $J^*(A; B)$ but with the subsets S and T in the defining sum having size smaller than q , i.e.

$$(25) \quad J_q^*(A; B) = \sum_{\substack{S \subset A, T \subset B \\ |S|=|T| < q}} \dots$$

Then, if the total support of Φ is limited to any number smaller than $2q$, then Theorem 4 holds with all the J^* replaced by J_q^* .

Theorem 5. *Let $\delta > 0$ and suppose that $F(x_1, \dots, x_n)$ satisfies (20). Then,*

$$(26) \quad \begin{aligned} & \int_{U(N)} \sum_{j_1, \dots, j_n} F(\theta_{j_1}, \dots, \theta_{j_n}) dX \\ &= \frac{1}{(2\pi i)^n} \sum_{\substack{K+L+M=\{1, \dots, n\} \\ |K|=|L|}} (-1)^{|L|+|M|} N^{|M|} \\ & \quad \times \int_{(\delta)^{|K|}} \int_{(-\delta)^{|L|}} \int_{(0)^{|M|}} J_q^*(z_K; -z_L) F(iz_1, \dots, iz_n) dz_1 \dots dz_n \end{aligned}$$

4. THE SPECIAL CASE $q = 1$

If $q = 1$, as in the theorem of Rudnick and Sarnak, then the sets S and T in the sum defining $J_1^*(A; B)$ are both empty. We have

$$(27) \quad J_1^*(A; B) = \sum_{\substack{A+B= \\ U_1+\dots+U_Q \\ |U_q|\leq 2}} \prod_{q=1}^Q H_{\emptyset, \emptyset}(U_q),$$

We have

$$(28) \quad H_{\emptyset, \emptyset}(U) = \begin{cases} \left(\frac{z'}{z}\right)'(\alpha + \beta) & \text{if } U = \{\alpha, \beta\} \text{ with } \begin{smallmatrix} \alpha \in A, \\ \beta \in B \end{smallmatrix} \\ 0 & \text{otherwise.} \end{cases}$$

This gives, when $A = \{\alpha_1, \dots, \alpha_K\}$, $B = \{\beta_1, \dots, \beta_K\}$,

$$(29) \quad J_1^*(A; B) = \sum_{\sigma \in S_K} \prod_{k=1}^K \left(\frac{z'}{z}\right)'(\alpha_k + \beta_{\sigma(k)}).$$

Thus, we have

Theorem 6. *Let $\delta > 0$ and suppose that $F(x_1, \dots, x_n)$ satisfies (20) with $q = 1$. Then,*

$$(30) \quad \begin{aligned} & \int_{U(N)} \sum_{j_1, \dots, j_n} F(\theta_{j_1}, \dots, \theta_{j_n}) dX \\ &= \frac{1}{(2\pi i)^n} \sum_{\substack{K+L+M=\{1, \dots, n\} \\ |K|=|L|}} (-1)^{|L|+|M|} N^{|M|} \\ & \quad \times \int_{(\delta)^{|K|}} \int_{(-\delta)^{|L|}} \int_{(0)^{|M|}} \sum_{\sigma \in S_K} \prod_{j=1}^K \left(\frac{z'}{z}\right)'(z_{k_j} - z_{\ell_{\sigma(j)}}) F(iz_1, \dots, iz_n) dz \end{aligned}$$

where

$$(31) \quad K = \{k_1, \dots, k_K\} \quad \text{and} \quad L = \{\ell_1, \dots, \ell_L\}.$$

5. COMPARISON WITH RUDNICK-SARNAK

We now put the formula from the last section into the form of Theorem 3.1 of [RuSa]. For ease of subscripting, consider one particular term, denoted (with an abuse of notation that hopefully won't cause confusion) $K = \{1, 2, \dots, K\}$, $L = \{K+1, \dots, 2K\}$ and σ is the identity permutation. Let

$$I := \int_{(\delta)^K} \int_{(-\delta)^K} \int_{(0)^{n-2K}} \prod_{k=1}^K \left(\frac{z'}{z}\right)'(z_k - z_{K+k}) F(iz_1, \dots, iz_n) dz_1 \dots dz_n.$$

We write

$$F(iz_1, \dots, iz_n) = h_1 \left(\frac{iz_1}{T} \right) \dots h_n \left(\frac{iz_n}{T} \right) \int_{\mathbf{R}^n} \Phi(\xi_1, \dots, \xi_n) \\ \times \delta(\xi_1 + \dots + \xi_n) e \left(-\frac{iNz_1\xi_1}{2\pi} - \dots - \frac{iNz_n\xi_n}{2\pi} \right) d\xi_1 \dots d\xi_n.$$

We integrate the z_r variables with $2K < r \leq 2n$ using equation (4) to get

$$\int_{(0)^{n-2K}} e \left(-\frac{iNz_{2K+1}\xi_{2K+1}}{2\pi} - \dots - \frac{iNz_n\xi_n}{2\pi} \right) h_{2K+1} \left(\frac{iz_{2K+1}}{T} \right) \dots h_n \left(\frac{iz_n}{T} \right) dz_{2K+1} \dots dz_n \\ = \prod_{j=2K+1}^n \int_{(0)} h_j \left(\frac{iz_j}{T} \right) e \left(\frac{iNz_j\xi_j}{2\pi} \right) dz_j = \prod_{j=2K+1}^n (-2\pi iT g_j(NT\xi_j)).$$

This gives

$$I = \int_{\mathbf{R}^n} \Phi(\xi_1, \dots, \xi_n) \delta(\xi_1 + \dots + \xi_n) \prod_{j=2K+1}^n (-2\pi iT g_j(NT\xi_j)) \int_{(\delta)^K} \int_{(-\delta)^K} \prod_{k=1}^K \\ \left(e \left(\frac{-iNz_k\xi_k}{2\pi} \right) h_k \left(\frac{iz_k}{T} \right) e \left(-\frac{iNz_{K+k}\xi_{K+k}}{2\pi} \right) \right. \\ \left. \times h_{K+k} \left(\frac{iz_{K+k}}{T} \right) \left(\frac{z'}{z} \right)' (z_k - z_{K+k}) \right) dz_1 \dots dz_{2K} d\xi.$$

Now we move the paths of integration of the z_k with $1 \leq k \leq K$. If $\xi_k > 0$ we move the path to the left across the double pole at z_{K+k} ; if $\xi_k < 0$ then we move the path far to the right. We use the fact that

$$\operatorname{Res}_{z_k=z_{K+k}} \frac{f_1(z_k)f_2(z_{K+k})}{(z_k - z_{K+k})^2} = f_1'(z_{K+k})f_2(z_{K+k}).$$

In this way we get

$$I = (1 + O(1/T)) (2\pi i)^K \int_{\substack{\mathbf{R}^n \\ \xi_j > 0, j \leq K}} \Phi(\xi_1, \dots, \xi_n) \delta(\xi_1 + \dots + \xi_n) \prod_{j=2K+1}^n (-2\pi iT g_j(NT\xi_j)) \\ \times \int_{(-\delta)^K} \prod_{k=1}^K \left(N\xi_k e \left(\frac{-iNz_{K+k}(\xi_k + \xi_{K+k})}{2\pi} \right) h_k \left(\frac{iz_{K+k}}{T} \right) h_{K+k} \left(\frac{iz_{K+k}}{T} \right) dz_{K+k} \right) d\xi.$$

We compute, for example,

$$\int_{(-\delta)} e \left(-\frac{iNz(\xi_1 + \xi_2)}{2\pi} \right) h_1 \left(\frac{iz}{T} \right) h_2 \left(\frac{iz}{T} \right) dz = -2\pi iT \int_{\mathbf{R}} g_1(u) g_2(-u + NT(\xi_1 + \xi_2)) du$$

Thus,

$$I = (2\pi i N)^K \int_{\substack{\mathbf{R}^n \\ \xi_j > 0, j \leq K}} \xi_1 \dots \xi_K \Phi(\xi_1, \dots, \xi_n) \delta(\xi_1 + \dots + \xi_n) \prod_{j=2K+1}^n (-2\pi i T g_j(NT\xi_j)) \\ \times \prod_{k=1}^K \left(-2\pi i T \int_{\mathbf{R}} g_k(u_k) g_{K+k}(-u_k + TN(\xi_k + \xi_{K+k})) du_k \right) d\xi (1 + O(\frac{1}{T})).$$

We make the changes of variables

$$\begin{aligned} y_j &= NT\xi_j && \text{for } 2K+1 \leq j \leq n \\ y_k &= u_k && \text{for } 1 \leq k \leq K \\ y_{K+k} &= -u_k + NT(\xi_k + \xi_{K+k}) && \text{for } 1 \leq k \leq K. \end{aligned}$$

The last substitution implies that

$$(32) \quad \xi_{K+k} = -\xi_k + \frac{(y_{K+k} + u_k)}{NT}.$$

Also, the condition $\sum_{j=1}^n \xi_j = 0$ implied by the delta-function translates to $\sum_{j=1}^n y_j = 0$. We have

$$I = \frac{(2\pi i N)^K (-2\pi i T)^{n-K}}{(NT)^{n-K-1}} \int_{\substack{\mathbf{R}^n \\ \sum y_j = 0}} \prod_{j=1}^n g_j(y_j) \int_{\xi_k > 0, 1 \leq k \leq K} \xi_1 \dots \xi_K \\ \times \Phi(\xi_1, \dots, \xi_K, -\xi_1 + \frac{y_{K+1} + y_1}{NT}, \dots, -\xi_K + \frac{(y_{2K} + y_K)}{NT}, \frac{y_{n-2K}}{NT}, \dots, \frac{y_n}{NT}) \\ \times (1 + O(\frac{1}{T})) d\xi_1 \dots d\xi_K dy_1 \dots dy_n.$$

Employing the Taylor expansion of Φ , just as in [RuSa], we have

$$\frac{I}{NT} = N^{2K-n} (2\pi i)^n (-1)^{n-K} \int_{\substack{\mathbf{R}^n \\ \sum y_j = 0}} \prod_{j=1}^n g_j(y_j) dy \int_{\substack{\mathbf{R}^K \\ \xi_k > 0, 1 \leq k \leq K}} \xi_1 \dots \xi_K \\ \times \Phi(\xi_1, \dots, \xi_K, -\xi_1, \dots, -\xi_K, 0, \dots, 0) d\xi_1 \dots d\xi_K (1 + O(1/T)).$$

Since

$$(33) \quad \int_{\substack{\mathbf{R}^n \\ \sum y_j = 0}} \prod_{j=1}^n g_j(y_j) dy = \int_{\mathbf{R}^{n-1}} \prod_{j=1}^{n-1} g_j(y_j) \frac{1}{2\pi} \int_{\mathbf{R}} h_n(t) \exp(-it(\sum_{j=1}^{n-1} y_j)) dt dy$$

$$(34) \quad = \frac{1}{2\pi} \int_{\mathbf{R}} \prod_{j=1}^n h_j(t) dt = \frac{\kappa(\mathbf{h})}{2\pi},$$

we have

$$\frac{I}{NT} = N^{2K-n} (2\pi i)^n (-1)^{n-K} \frac{\kappa(\mathbf{h})}{2\pi} \int_{\substack{\mathbf{R}^K \\ \xi_k > 0, 1 \leq k \leq K}} \xi_1 \dots \xi_K \\ \times \Phi(\xi_1, \dots, \xi_K, -\xi_1, \dots, -\xi_K, 0, \dots, 0) d\xi_1 \dots d\xi_K (1 + O(1/T)).$$

More generally, for $K = \{k_1, \dots, k_{|K|}\}$, $L = \{\ell_1, \dots, \ell_{|K|}\}$ and $\sigma \in S_{|K|}$,

$$\begin{aligned} I(K, L, \sigma) : &= \int_{(\delta)^{|K|}} \int_{(-\delta)^{|L|}} \int_{(0)^{|M|}} \prod_{j=1}^K \left(\frac{z'}{z} \right)' (z_{k_j} - z_{\ell_{\sigma(j)}}) F(iz_1, \dots, iz_n) dz \\ &\sim (NT) N^{2|K|-n} (2\pi i)^n (-1)^{n-|K|} \frac{\kappa(\mathbf{h})}{2\pi} \int_{\xi_{k_j} > 0} \xi_{k_1} \dots \xi_{k_{|K|}} \Phi \left(\sum_{j=1}^{|K|} \xi_{k_j} \mathbf{e}_{k_j, \ell_{\sigma(j)}} \right) d\xi. \end{aligned}$$

We insert this into Theorem 6 and have

$$\begin{aligned} &\int_{U(N)} \sum_{j_1, \dots, j_n} F(\theta_{j_1}, \dots, \theta_{j_n}) dX \\ &= \frac{1}{(2\pi i)^n} \sum_{\substack{K+L+M=\{1, \dots, n\} \\ |K|=|L|}} (-1)^{|L|+|M|} N^{|M|} N^{2|K|-n} (2\pi i)^n (-1)^{n-|K|} \\ &\quad \times \frac{\kappa(\mathbf{h}) NT}{2\pi} \int_{\xi_{k_j} > 0} \xi_{k_1} \dots \xi_{k_{|K|}} \sum_{\sigma \in S_K} \Phi \left(\sum_{j=1}^{|K|} \xi_{k_j} \mathbf{e}_{k_j, \ell_{\sigma(j)}} \right) d\xi (1 + O(1/T)). \end{aligned}$$

Since $|M| = n - 2|K|$ the right-hand side simplifies to

$$\kappa(\mathbf{h}) \frac{NT}{2\pi} \sum_{\substack{K+L+M=\{1, \dots, n\} \\ |K|=|L|}} \sum_{\sigma \in S_{|K|}} \int_{\xi_{k_j} > 0} \xi_{k_1} \dots \xi_{k_{|K|}} \Phi \left(\sum_{j=1}^{|K|} \xi_{k_j} \mathbf{e}_{k_j, \ell_{\sigma(j)}} \right) d\xi + O(N).$$

This proves Theorem 3.

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